

A calculation with a bi-orthogonal wavelet transformation *

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Abstract

We explore the use of bi-orthogonal basis for continuous wavelet transformations, thus relaxing the so-called admissibility condition on the analyzing wavelet. As an application, we determine the eigenvalues and corresponding radial eigenfunctions of the Hamiltonian of relativistic Hydrogen-like atoms.

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1 - Introduction

Wavelet transforms have been successfully used in Mathematics, Physics and Engineering [1, 2, 3]. In particular, in the context of Quantum Mechanics, continuous wavelet transforms have proved very useful, giving rise - for example - to entirely new approaches to problems with spherical symmetry. For nonrelativistic Hydrogen-like atoms, an adequate choice of the analyzing wavelet reduces the radial Schrödinger equation to a first order differential equation, and the analyticity of wavelet coefficients leads, in a straightforward manner, to the determination of the eigenvalues and their corresponding eigenfunctions [4].

In this context, the selection of an analyzing wavelet is constrained by the "admissibility condition", which guarantees the existence of an inverse transform [5, 4]. On the other hand, bi-orthogonal basis have been introduced in the context of discrete [6, 7] as well as continuous [8] transforms.

However, in some cases, computational convenience may suggest that the most adequate "analyzing wavelet" be a non-admissible and even a non-square-integrable function. This is the case, for example, for relativistic Hydrogen-like atoms, as we will see later.

It is the aim of this paper to extend the wavelet analysis to such situations, where it is not possible to construct an orthogonal continuous basis of \mathbf{L}^2 , via the " $ax + b$ " transform of the analyzing wavelet. In order to get an invertible transformation, we will rather restrict ourselves to a subspace of \mathbf{L}^2 (containing the bounded eigenstates of the Hamiltonian to be treated) and make use of bi-orthogonal continuous basis.

In section 2, we consider the space where this wavelet transform is well defined and some of its properties. We propose sufficient conditions for a function to belong to the space of wavelet coefficients. Such conditions are satisfied by the transformed eigenfunctions of the relativistic Hamiltonian to be treated later.

In section 3, the radial Dirac equation is solved for relativistic Hydrogen-like atoms. As in the nonrelativistic case presented in reference [4], the analyticity of the space of coefficients is shown to determine the spectrum. Moreover, the asymptotic behaviour of functions in this space allows for a determination of the associated eigenfunctions.

Finally, in section 4, we present some comments and conclusions.

2 - The transformation

Let us consider a function $\psi^\gamma(q)$, solution of

$$\left(\frac{d}{dq} + \frac{2-\gamma}{q}\right) \psi^\gamma(q) = -\psi^\gamma(q), \quad (1)$$

with $q \in [0, \infty)$:

$$\psi^\gamma(q) = q^{\gamma-2} e^{-q}. \quad (2)$$

For $\gamma > 1$, $\psi^\gamma(q)$ is an admissible wavelet [5]. So, by considering its "ax + b" group transformation,

$$\psi_z^\gamma(q) = a^{3/2} e^{ibq} \left[(aq)^{\gamma-2} e^{-aq} \right], \text{ with } z = b + ia \text{ and } a > 0, \quad (3)$$

a continuous orthogonal basis of $\mathbf{L}^2(\mathbf{R}^+, q^2 dq)$ can be defined as $\{\psi_z^\gamma(q)\}$ (For definiteness, we will consider the radial part of three dimensional problems).

Therefore, the wavelet coefficient of a function $f(q) \in \mathbf{L}^2(\mathbf{R}^+, q^2 dq)$ is given by:

$$(\psi_z^\gamma(q), f(q)) = a^{\gamma-1/2} F(\bar{z}), \quad (4)$$

where

$$F(\bar{z}) = \mathcal{L}^\gamma(f(q))(\bar{z}) = \int_0^\infty dq e^{-i\bar{z}q} q^\gamma f(q) \quad (5)$$

is an analytic function of the variable \bar{z} in the lower half-plane. One then has the reconstruction formula:

$$f(q) = \frac{2^{2\gamma-2}}{2\pi\Gamma(2\gamma-2)} \int_{\{Im z > 0\}} d\mu_L(z) (\psi_z^\gamma(q), f(q)) \psi_z^\gamma(q) \quad (6)$$

with $d\mu_L(z)$ the left invariant measure of the "ax + b" group:

$$d\mu_L(z) = \frac{da db}{a^2}. \quad (7)$$

Moreover, the following equality holds:

$$\int_0^\infty dq q^2 |f(q)|^2 = \frac{2^{2\gamma-2}}{2\pi\Gamma(2\gamma-2)} \int_{\{Im z > 0\}} d\mu_L(z) (Im z)^{2\gamma-1} |F(\bar{z})|^2, \quad (8)$$

which shows that $F(\bar{z})$ belongs to a Bergman space $\mathcal{B}_{2\gamma-1}$ (see reference [4]).

Now, for $1/2 < \gamma < 1$, the analyzing wavelet chosen is not an admissible one [5]. So, in this range, it is not possible to construct an orthogonal basis leading to the reconstruction formula (6). Moreover, for $0 < \gamma \leq 1/2$, $\psi^\gamma(q)$ is not even a square integrable function, so that the integral in equation (5) doesn't exist for an arbitrary $f(q) \in \mathbf{L}^2(\mathbf{R}^+, q^2 dq)$.

In what follows, we will be interested in showing that it is still possible to use the transform in equation (5) for solving an eigenvalue problem, provided certain regularity conditions are satisfied by its solutions. We will also analyze which properties of an authentic wavelet transform do still hold in such a situation.

To this end, we will introduce a bi-orthogonal continuous basis. That is, we will make use of different functions in the process of analysis and later reconstruction:

$$f(q) = \int_{\{Im\, z > 0\}} d\mu_L(z) (\psi_z^\gamma(q), f(q)) \chi_z^\gamma(q), \quad (9)$$

where $\chi_z^\gamma(q)$ is obtained - through the action of the group " $ax + b$ " - from a function $\chi^\gamma(q)$ satisfying:

$$\int_0^\infty dq\, q\, \psi^\gamma(q)^* \chi^\gamma(q) = \frac{1}{2\pi}. \quad (10)$$

Then, the following Lemmas hold:

Lemma 1 *Let $f(q) \in \mathbf{L}_{loc}^1(\mathbf{R}^+, q^\gamma dq) \cap \mathbf{L}^2((1, \infty), dq)$, with $0 < \gamma < 1$, and consider $F(\bar{z})$ as defined in equation(5). Then:*

a) *$F(\bar{z})$ is an analytic function in the half-plane $Im\, \bar{z} < 0$. Moreover,*

$$F(\bar{z}) \xrightarrow{|Re\, z| \rightarrow \infty} 0, \text{ with } Im\, z = a > 0, \text{ and } F(\bar{z}) \xrightarrow{Im\, z \rightarrow \infty} 0.$$

b) *If $f(q) \sim q^{\alpha-1}$ ($\alpha \geq 0$) for $q \sim 0$ and $f(q)$ is bounded when $q \rightarrow \infty$, then \mathcal{L}^γ transforms the operator qd/dq into the operator $-\bar{z}\partial/\partial\bar{z} - (\gamma + 1)$.*

c) *If $f(q) \in \mathbf{L}^2(\mathbf{R}^+, q^2 dq)$ then: $\partial_{\bar{z}} F(\bar{z}) \in \mathcal{B}_{2(\gamma+1)-1}$, and*

$$\int_0^\infty dq\, q^2 |f(q)|^2 =$$

$$\frac{2^{2(\gamma+1)-2}}{2\pi\Gamma(2(\gamma+1)-2)} \int_{Im z > 0} d\mu_L(z) (Im z)^{2(\gamma+1)-1} |\partial_{\bar{z}} F(\bar{z})|^2. \quad (11)$$

Proof:

a) The function

$$F(\bar{z}) = F(b - ia) = \int_0^\infty dq e^{-ibq} q^\gamma f(q) e^{-aq} \quad (12)$$

is the Fourier transform of $q^\gamma f(q) e^{-aq} \in \mathbf{L}^1(\mathbf{R}^+, dq)$. So:

$$F(\bar{z}) \xrightarrow{|Re \bar{z}| \rightarrow \infty} 0, \text{ for } Im \bar{z} = -a < 0. \quad (13)$$

The analyticity of $F(\bar{z})$ and the fact that $F(\bar{z}) \rightarrow 0$ for $Im \bar{z} \rightarrow -\infty$ are direct consequences of its definition (see equation(12)), since $q^\gamma f(q) e^{-aq} \in \mathbf{L}^1(\mathbf{R}^+, dq)$ for $a > 0$.

b) Now,

$$\begin{aligned} & \int_\varepsilon^\Lambda dq e^{-i\bar{z}q} q^\gamma \left[q \frac{d}{dq} f(q) \right] = \\ & e^{-i\bar{z}q} q^{\gamma+1} f(q) \Big|_\varepsilon^\Lambda - \int_\varepsilon^\Lambda dq \frac{d}{dq} \left[e^{-i\bar{z}q} q^{\gamma+1} \right] f(q) \\ & \xrightarrow[\varepsilon \rightarrow 0]{\Lambda \rightarrow \infty} - [\bar{z} \partial_{\bar{z}} + \gamma + 1] \int_0^\infty dq e^{-i\bar{z}q} q^\gamma f(q), \end{aligned} \quad (14)$$

since the integrated term vanishes under the assumption made on the behavior of $f(q)$, and $e^{-i\bar{z}q} q^{\gamma+1} f(q) \in \mathbf{L}^1(\mathbf{R}^+, dq)$.

c) Notice that:

$$\partial_{\bar{z}} F(\bar{z}) = -i \int_0^\infty dq e^{-i\bar{z}q} q^{\gamma+1} f(q) = \mathcal{L}^{\gamma+1}(f(q))(\bar{z}) \quad (15)$$

is the analytic factor of the wavelet coefficient of $f(q)$ with respect to the wavelet $\psi_z^{\gamma+1}(q) \in \mathbf{L}^2(\mathbf{R}^+, q^2 dq)$, wich is admissible (since $\gamma + 1 > 1$). Then, from equation (8) we immediately get equation (11). \square

Lemma 2 *Let $F(\bar{z})$ be an analytic function in the half-plane $\{Im \bar{z} < 0\}$, with an asymptotic behaviour given by:*

$$F(\bar{z}) = C_0 (\bar{z} - \bar{z}_0)^{-(\gamma+\alpha)} + C_1 (\bar{z} - \bar{z}_0)^{-(\gamma+\alpha+1)} + G(\bar{z}), \quad (16)$$

where $Im \bar{z}_0 > 0$ and $|G(\bar{z})| \leq K |\bar{z}|^{-(\gamma+\alpha+2)}$ is locally bounded in the half-plane $Im \bar{z} \leq 0$ (C_0 , C_1 and K are constants). Then:

a) $(Im \bar{z})^{\gamma-1/2} F(\bar{z})$ is the wavelet coefficient of a function $f(q) \in \mathbf{L}_{loc}^1(\mathbf{R}^+, q^\gamma dq) \cap \mathbf{L}^2((1, \infty), dq)$, given by:

$$f(q) = \int_0^\infty \frac{da}{a^2} \int_{-\infty}^\infty db (Im z)^{\gamma-1/2} F(\bar{z}) \chi_z^\gamma(q), \quad (17)$$

with $\bar{z} = b - ia$.

b) If $\partial_{\bar{z}} F(\bar{z}) \in \mathcal{B}_{2(\gamma+1)-1}$, then $f(q) \in \mathbf{L}^2(\mathbf{R}^+, q^2 dq)$.

c) If $|\bar{z} \partial_{\bar{z}} G(\bar{z})| \leq K' |\bar{z}|^{-(\gamma+\alpha+2)}$, and is locally bounded in the half-plane $Im \bar{z} \leq 0$ (K' is a constant), then $\bar{z} \partial_{\bar{z}} F(\bar{z}) = \mathcal{L}^\gamma(h(q))(\bar{z})$, where

$$h(q) = -\left(q \frac{d}{dq} + \gamma + 1\right) f(q). \quad (18)$$

Proof:

a) In the first place, notice that $(\bar{z} - \bar{z}_0)^{-(\gamma+\alpha)}$, with $\alpha \geq 0$, is the analytic factor in the wavelet coefficient corresponding to the function $f_0(q) = C_0 \left(i^{(\gamma+\alpha)} / \Gamma(\gamma + \alpha) \right) q^{\alpha-1} e^{i\bar{z}_0 q} \in \mathbf{L}_{loc}^1(\mathbf{R}^+, q^\gamma dq) \cap \mathbf{L}^2((1, \infty), dq)$. In fact,

$$\begin{aligned} \mathcal{L}^\gamma[q^{\alpha-1} e^{i\bar{z}_0 q}](\bar{z}) &= \mathcal{F}[q^{\gamma+\alpha-1} e^{-(a-a_0)q}](b - b_0) \\ &= \int_0^\infty dq q^{\gamma+\alpha-1} e^{-i(\bar{z}-\bar{z}_0)q} = \frac{\Gamma(\gamma + \alpha)}{[i(\bar{z} - \bar{z}_0)]^{\gamma+\alpha}}. \end{aligned} \quad (19)$$

So:

$$\begin{aligned} \int_{\{Im z > 0\}} d\mu_L(z) (Im z)^{\gamma-1/2} (\bar{z} - \bar{z}_0)^{-(\gamma+\alpha)} \chi_z^\gamma(q) &= \\ \int_0^\infty da a^{\gamma-1} \chi^\gamma(aq) 2\pi \mathcal{F}^{-1}[(\bar{z} - \bar{z}_0)^{-(\gamma+\alpha)}](q) &= \frac{i^{\gamma+\alpha}}{\Gamma(\gamma + \alpha)} q^{\alpha-1} e^{i\bar{z}_0 q}. \end{aligned} \quad (20)$$

(Notice that the integral in the first member is conditionally convergent). A similar result holds, changing α into $\alpha + 1$, for the second term in equation (16), which is the analytic factor in the wavelet coefficient of $f_1(q) = C_1 \left(i^{(\gamma+\alpha+1)} / \Gamma(\gamma + \alpha + 1) \right) q^\alpha e^{i\bar{z}_0 q} \in \mathbf{L}_{loc}^1(\mathbf{R}^+, q^\gamma dq) \cap \mathbf{L}^2((1, \infty), dq)$.

As regards $G(\bar{z})$, under the assumptions made, it belongs to the Bergman space $B_{2\gamma+1}$, since

$$\int_{\{Im z > 0\}} d\mu_L(z) (Im z)^{2(\gamma+1)-1} |G(\bar{z})|^2 < \infty, \quad (21)$$

as can be easily verified: For example,

$$\int_0^1 \int_{-1}^1 |G(\bar{z})|^2 a^{2\gamma-1} db da < \infty, \quad (22)$$

since $|G(\bar{z})|$ is locally bounded.

Moreover, $\{\psi_z^{\gamma+1}\}$ is an orthogonal wavelet basis of $\mathbf{L}^2(\mathbf{R}^+, q^2 dq)$, which defines a bijection onto $B_{2\gamma+1}$ (see reference [4]). Then, $g(q) \in \mathbf{L}^2(\mathbf{R}^+, q^2 dq)$ exists such that:

$$G(\bar{z}) = \int_0^\infty dq q^{\gamma+1} e^{-i\bar{z}q} g(q) = \mathcal{L}^{\gamma+1}(g(q))(\bar{z}), \quad (23)$$

or, equivalently:

$$G(\bar{z}) = \int_0^\infty dq q^\gamma e^{-i\bar{z}q} f_2(q) = \mathcal{F}[q^\gamma f_2(q) e^{-aq}] = \mathcal{L}^\gamma(f_2(q))(\bar{z}), \quad (24)$$

with $f_2(q) = qg(q) \in \mathbf{L}^2(\mathbf{R}^+, dq)$ and, therefore, $f_2(q) \in \mathbf{L}_{loc}^1(\mathbf{R}^+, q^\gamma dq) \cap \mathbf{L}^2((1, \infty), dq)$.

Finally

$$\begin{aligned} & \int_{\{Im z > 0\}} d\mu_L(z) (Im z)^{\gamma-1/2} G(\bar{z}) \chi_z^\gamma(q) = \\ & \int_0^\infty da a^{\gamma-1} \chi^\gamma(aq) \int_{-\infty}^\infty db G(b - ia) e^{ibq} = \\ & 2\pi \int_0^\infty da a^{\gamma-1} \chi^\gamma(aq) q^\gamma f_2(q) e^{-aq} = f_2(q), \end{aligned} \quad (25)$$

where use has been made of the fact that $G(\bar{z})$ is the Fourier transform of a square integrable function.

b) Let us suppose that $\partial_{\bar{z}}F(\bar{z}) \in B_{2\gamma+1}$; then a function $h(q) \in \mathbf{L}^2(\mathbf{R}^+, q^2 dq)$ exists such that:

$$\partial_{\bar{z}}F(\bar{z}) = \int_0^\infty dq q^{\gamma+1} h(q) e^{-i\bar{z}q} = \mathcal{L}^{\gamma+1}(h(q))(\bar{z}). \quad (26)$$

Moreover, from a), we know that:

$$\begin{aligned} \partial_{\bar{z}}F(\bar{z}) &= \partial_{\bar{z}} \int_0^\infty dq q^\gamma [f_0(q) + f_1(q) + f_2(q)] e^{-i\bar{z}q} \\ &= -i \int_0^\infty dq q^{\gamma+1} [f_0(q) + f_1(q) + f_2(q)] e^{-i\bar{z}q}, \end{aligned} \quad (27)$$

since the last integral is absolutely convergent.

Then, from a) (with $\gamma \rightarrow \gamma + 1$), we conclude that $f_0(q) + f_1(q) + f_2(q) = f(q) = ih(q) \in \mathbf{L}^2(\mathbf{R}^+, q^2 dq)$.

c) In the first place, we will consider, for $\alpha \geq 0$:

$$\begin{aligned} &\int_{\{Im z > 0\}} d\mu_L(z) (Im z)^{\gamma-1/2} [\bar{z} \partial_{\bar{z}}(\bar{z} - \bar{z}_0)^{-(\gamma+\alpha)}] \chi_z^\gamma(q) = \\ &-(\gamma+\alpha) \int_{\{Im z > 0\}} d\mu_L(z) (Im z)^{\gamma-1/2} \left[\frac{1}{(\bar{z} - \bar{z}_0)^{\gamma+\alpha}} + \frac{\bar{z}_0}{(\bar{z} - \bar{z}_0)^{\gamma+\alpha+1}} \right] \chi_z^\gamma(q). \end{aligned} \quad (28)$$

From equation (20), the previous expresion reduces to:

$$\begin{aligned} &-(\gamma + \alpha) \left[\frac{i^{\gamma+\alpha}}{\Gamma(\gamma + \alpha)} q^{\alpha-1} e^{i\bar{z}_0 q} + \bar{z}_0 \frac{i^{\gamma+\alpha+1}}{\Gamma(\gamma + \alpha + 1)} q^{(\alpha+1)-1} e^{i\bar{z}_0 q} \right] = \\ &-(q \frac{d}{dq} + \gamma + 1) \frac{i^{\gamma+\alpha}}{\Gamma(\gamma + \alpha)} q^{\alpha-1} e^{i\bar{z}_0 q}, \end{aligned} \quad (29)$$

which proves the statement for the first two terms in equation (16).

As concerns the third one:

$$\begin{aligned} &\int_{\{Im z > 0\}} d\mu_L(z) (Im z)^{\gamma-1/2} [\bar{z} \partial_{\bar{z}} G(\bar{z})] \chi_z^\gamma(q) = \\ &\int_0^\infty da a^{\gamma-1} \chi^\gamma(aq) \left[\int_{-\infty}^\infty db (b - ia) \partial_b G(b - ia) e^{ibq} \right] = \end{aligned}$$

$$\int_0^\infty da a^{\gamma-1} \chi^\gamma(aq) \left[- \int_{-\infty}^\infty db G(b-ia)(1+aq+ibq)e^{ibq} \right], \quad (30)$$

where use has been made of the asymptotic behaviour of $G(\bar{z})$ when integrating by parts.

Notice that the integral between brackets in equation (30) is absolutely convergent, so that:

$$\begin{aligned} & \left[\int_{-\infty}^\infty db G(b-ia)(1+aq+ibq)e^{ibq} \right] = \\ & \left(1 + aq + q \frac{d}{dq} \right) \int_{-\infty}^\infty db G(b-ia)e^{ibq}. \end{aligned} \quad (31)$$

Now, since $G(\bar{z}) \in B_{2\gamma+1}$, one has:

$$\begin{aligned} \mathcal{F}^{-1}[G(b-ia)](q) &= \frac{1}{2\pi} \int_{-\infty}^\infty db G(b-ia)e^{ibq} \\ &= q^\gamma g(q) e^{-aq}, \end{aligned} \quad (32)$$

with $g(q) \in \mathbf{L}^2(\mathbf{R}^+, dq)$. Therefore:

$$\begin{aligned} & \left(1 + aq + q \frac{d}{dq} \right) 2\pi q^\gamma g(q) e^{-aq} = \\ & 2\pi q^\gamma e^{-aq} \left(1 + \gamma + q \frac{d}{dq} \right) g(q) \end{aligned} \quad (33)$$

and

$$\begin{aligned} & \int_{\{Im z > 0\}} d\mu_L(z) (Im z)^{\gamma-1/2} [\bar{z} \partial_{\bar{z}} G(\bar{z})] \chi_z^\gamma(q) = \\ & \int_0^\infty da a^{\gamma-1} \chi^\gamma(aq) \left[-2\pi q^\gamma e^{-aq} \left(1 + \gamma + q \frac{d}{dq} \right) g(q) \right] \\ & = - \left(1 + \gamma + q \frac{d}{dq} \right) g(q), \end{aligned} \quad (34)$$

which completes the proof. \square

For $0 < \gamma < 1$, the space of wavelet coefficients that appears in Lemma 1.a) consists of functions $(Im z)^{\gamma-1/2} F(\bar{z})$, where $F(\bar{z})$ is analytic for $Im \bar{z} < 0$, vanishes for $\bar{z} \rightarrow \infty$ and is such that $\partial_{\bar{z}} F(\bar{z})$ belongs to $\mathcal{B}_{2(\gamma+1)-1}$. This space of coefficients corresponds to the transforms of functions in $\mathbf{L}_{loc}^1(\mathbf{R}^+, q^\gamma dq) \cap \mathbf{L}^2((1, \infty), dq)$.

Now, we introduce the linear space \mathcal{A}_γ of functions $F(\bar{z})$, analytic in the half plane $Im \bar{z} < 0$, vanishing for $\bar{z} \rightarrow \infty$ and such that $\partial_{\bar{z}} F(\bar{z}) \in \mathcal{B}_{2(\gamma+1)-1}$. Obviously, it is a pre-Hilbert space with respect to the scalar product:

$$\langle F|G \rangle_{\mathcal{A}_\gamma} = \int_{Im z > 0} d\mu_L(z) (Im z)^{2(\gamma+1)-1} \partial_{\bar{z}} F(\bar{z})^* \partial_{\bar{z}} G(\bar{z}). \quad (35)$$

Lemma 3 *The transformation \mathcal{L}^γ , defined in equation (5) for $0 < \gamma < 1$, maps a dense subspace of $\mathbf{L}^2(\mathbf{R}^+, q^2 dq)$ into a dense subspace of the pre-Hilbert space \mathcal{A}_γ , preserving the norm.*

Proof:

Notice, in the first place, that the complete set of functions of $\mathbf{L}^2(\mathbf{R}^+, q^2 dq)$ given by $\{\psi_n(q) = q^{\alpha-1+n} e^{-q}, n = 0, 1, 2, \dots\}$, with $0 \leq \alpha < 1$, is contained in $\mathbf{L}_{loc}^1(\mathbf{R}^+, q^\gamma dq) \cap \mathbf{L}^2((1, \infty), dq)$. So, \mathcal{L}^γ is defined on a dense subspace of $\mathbf{L}^2(\mathbf{R}^+, q^2 dq)$. Moreover:

$$\begin{aligned} \mathcal{L}^\gamma(\psi_n(q))(\bar{z}) &= \int_0^\infty dq q^{\gamma+\alpha-1+n} e^{-q} e^{-i\bar{z}q} \\ &= \Gamma(\gamma + \alpha + n) [i(\bar{z} - i)]^{-(\gamma+\alpha+n)}. \end{aligned} \quad (36)$$

Now, the set $\{\mathcal{L}^\gamma(\psi_n(q)), n = 0, 1, 2, \dots\}$ is complete in \mathcal{A}_γ , since

$$i\partial_{\bar{z}} \mathcal{L}^\gamma(\psi_n(q))(\bar{z}) = \int_0^\infty dq q^{\gamma+\alpha+n} e^{-q} e^{-i\bar{z}q} = \mathcal{L}^{\gamma+1}(\psi_n(q))(\bar{z}), \quad (37)$$

and because of the isometry established by the wavelet transformation $\mathcal{L}^{\gamma+1}$ between the Hilbert spaces $\mathbf{L}^2(\mathbf{R}^+, q^2 dq)$ and $\mathcal{B}_{2(\gamma+1)-1}$ (see equation(8)).

Finally, for $f(q), g(q) \in \mathbf{L}^2(\mathbf{R}^+, q^2 dq) \cap (\mathbf{L}_{loc}^1(\mathbf{R}^+, q^\gamma dq) \cap \mathbf{L}^2((1, \infty), dq))$ we have,

$$\langle \mathcal{L}^\gamma(f(q))(\bar{z}) | \mathcal{L}^\gamma(g(q))(\bar{z}) \rangle_{\mathcal{A}_\gamma} =$$

$$\begin{aligned}
& \langle \mathcal{L}^{\gamma+1}(f(q))(\bar{z}) | \mathcal{L}^{\gamma+1}(g(q))(\bar{z}) \rangle_{\mathcal{B}_{2\gamma+1}} = \\
& \frac{2\pi\Gamma(2\gamma-2)}{2^{2\gamma-2}} (f, g)_{\mathbf{L}^2(\mathbf{R}^+, q^2 dq)} \cdot \square
\end{aligned} \tag{38}$$

3 - Relativistic Hydrogen-like atom

As an application of the results presented in the previous section, we proceed, in what follows, to the determination of the bounded eigenstates of the Hamiltonian of relativistic Hydrogen-like atoms.

As is well known [11], after eliminating angular variables through the SU(2) symmetry enjoyed by the problem at hand, the radial part of the eigenfunctions satisfies the following equations:

$$\begin{aligned}
\frac{df}{dr} + \frac{1+\chi}{r}f - \left(\varepsilon + m + \frac{\lambda}{r}\right)g &= 0 \\
\frac{dg}{dr} + \frac{1-\chi}{r}g - \left(\varepsilon - m + \frac{\lambda}{r}\right)f &= 0,
\end{aligned} \tag{39}$$

where m is the electron mass, and ε are the allowed eigenvalues, satisfying $|\varepsilon| < m$ for bounded states.

Moreover, $\lambda = N\alpha$ (with N the number of protons in the nucleus and $\alpha = 1/137$, the fine structure constant). In turn, χ is determined by the representation of SU(2) under study, and is given by:

$$\chi = \begin{cases} +(j+1/2), & \text{for } j = l - 1/2 \\ -(j+1/2), & \text{for } j = l + 1/2 \end{cases}, \tag{40}$$

with j the total angular momentum of the electron.

By defining:

$$0 \leq q = 2r\sqrt{m^2 - \varepsilon^2}, \tag{41}$$

equation (39) can be rewritten as:

$$\left(q \frac{d}{dq} + 1 + \chi\right) f(q) - \left(\frac{d}{2} \sqrt{\frac{m+\varepsilon}{m-\varepsilon}} + \lambda\right) g(q) = 0$$

$$\left(q \frac{d}{dq} + 1 - \chi\right) g(q) - \left(\frac{d}{2} \sqrt{\frac{m+\varepsilon}{m-\varepsilon}} - \lambda\right) f(q) = 0, \quad (42)$$

where $q f(q)$ and $q g(q)$ are square-integrable.

As it can be easily seen [11], for $q \rightarrow 0$, the solutions of equation (42) behave as:

$$f(q), g(q) \sim q^{-1+\sqrt{\chi^2-\lambda^2}}, \quad (43)$$

with $\chi^2 > \lambda^2$. So, $f(q), g(q) \in \mathbf{L}_{loc}^1(\mathbf{R}^+, q^\gamma dq)$, for $\gamma > 0$. The transformation discussed in the previous section can therefore be applied since $f(q)$ and $g(q)$ satisfy the requirements of Lemma 1.

Taking into account that the transformation is given by:

$$F(\bar{z}) = \mathcal{L}^\gamma(f(q))(\bar{z}) = \int_0^\infty dq e^{-i\bar{z}q} q^\gamma f(q), \quad (44)$$

it is easy to see that (Lemma 1):

$$\mathcal{L}^\gamma q \frac{d}{dq} = -\left(\bar{z} \frac{d}{d\bar{z}} + \gamma + 1\right) \mathcal{L}^\gamma, \quad (45)$$

and:

$$\mathcal{L}^\gamma q = i \frac{d}{d\bar{z}} \mathcal{L}^\gamma. \quad (46)$$

So, transforming equations (42), one gets:

$$\begin{aligned} \left(-\bar{z} \frac{d}{d\bar{z}} + \chi - \gamma\right) F(\bar{z}) - \left(\frac{i}{2} \sqrt{\frac{m+\varepsilon}{m-\varepsilon}} \frac{d}{d\bar{z}} + \lambda\right) G(\bar{z}) &= 0 \\ \left(-\bar{z} \frac{d}{d\bar{z}} - \chi - \gamma\right) G(\bar{z}) - \left(\frac{i}{2} \sqrt{\frac{m-\varepsilon}{m+\varepsilon}} \frac{d}{d\bar{z}} - \lambda\right) F(\bar{z}) &= 0. \end{aligned} \quad (47)$$

After some direct algebra, and calling

$$\Phi(\bar{z}) = \begin{pmatrix} F(\bar{z}) \\ G(\bar{z}) \end{pmatrix}, \quad (48)$$

equation (47) can be recast in the form:

$$\frac{d}{d\bar{z}} \Phi(\bar{z}) = -\frac{1}{2} \left\{ \frac{A' + B'}{\bar{z} - \frac{i}{2}} + \frac{A' - B'}{\bar{z} + \frac{i}{2}} \right\} \Phi(\bar{z}), \quad (49)$$

with:

$$A' = \begin{pmatrix} \gamma - \chi & \lambda \\ -\lambda & \gamma + \chi \end{pmatrix} \quad (50)$$

$$B' = \begin{pmatrix} \lambda \sqrt{\frac{m+\varepsilon}{m-\varepsilon}} & -(\chi + \gamma) \sqrt{\frac{m+\varepsilon}{m-\varepsilon}} \\ -(\gamma - \chi) \sqrt{\frac{m-\varepsilon}{m+\varepsilon}} & -\lambda \sqrt{\frac{m-\varepsilon}{m+\varepsilon}} \end{pmatrix}. \quad (51)$$

As is well known, the solution to equation (49) is given by:

$$\Phi(\bar{z}) = \mathbf{P} \exp \left\{ -\frac{1}{2} \int_{\bar{z}_0}^{\bar{z}} d\bar{z}' \left[\frac{A' + B'}{\bar{z}' - \frac{i}{2}} + \frac{A' - B'}{\bar{z}' + \frac{i}{2}} \right] \right\} \Phi(\bar{z}_0), \quad (52)$$

where \mathbf{P} means ordering over the path leading from \bar{z}_0 to \bar{z} .

Now, this expression can be greatly simplified through a judicious choice of γ : By taking ¹

$$\gamma = +\sqrt{\chi^2 - \lambda^2} > 0, \quad (53)$$

one has:

$$\begin{aligned} (A')^2 &= 2\gamma A' \quad , \quad A'B' = \frac{2\lambda\varepsilon}{\sqrt{m^2 - \varepsilon^2}} A', \\ (B')^2 &= \frac{2\lambda\varepsilon}{\sqrt{m^2 - \varepsilon^2}} B' \quad , \quad B'A' = 2\gamma B', \end{aligned} \quad (54)$$

and two new matrices can be defined as:

$$A = \frac{A' + B'}{-2\eta} \quad , \quad B = \frac{A' - B'}{-2\tilde{\eta}}, \quad (55)$$

where

$$\eta = -\gamma - \frac{\lambda\varepsilon}{\sqrt{m^2 - \varepsilon^2}}, \quad \tilde{\eta} = -\gamma + \frac{\lambda\varepsilon}{\sqrt{m^2 - \varepsilon^2}}. \quad (56)$$

¹Notice that, for $\chi^2 = (j + 1/2)^2 < 1 + \lambda^2$, $\gamma < 1$, and we are in the conditions of the Lemmas of Section 2.

So, the following relations hold:

$$\begin{aligned} A^2 &= A \quad , \quad AB = A, \\ B^2 &= B \quad , \quad BA = B. \end{aligned} \tag{57}$$

For this choice of γ it is easy to see that (52) reduces to:

$$\begin{aligned} \Phi(\bar{z}) - \Phi(\bar{z}_0) &= \\ \int_{\bar{z}_0}^{\bar{z}} d\bar{z}' \left(\frac{\eta A}{\bar{z}' - \frac{i}{2}} + \frac{\tilde{\eta} B}{\bar{z}' + \frac{i}{2}} \right) \left(\frac{\bar{z}' - \frac{i}{2}}{\bar{z}'_0 - \frac{i}{2}} \right)^\eta \left(\frac{\bar{z}' + \frac{i}{2}}{\bar{z}'_0 + \frac{i}{2}} \right)^{\tilde{\eta}} \Phi(\bar{z}_0). \end{aligned} \tag{58}$$

Determination of the spectrum

As discussed in Section 2, $\Phi(\bar{z})$ is an analytic function in the lower half-plane. So, its derivative:

$$\frac{d\Phi}{d\bar{z}} = \left(\frac{\eta A}{\bar{z}' - \frac{i}{2}} + \frac{\tilde{\eta} B}{\bar{z}' + \frac{i}{2}} \right) \left(\frac{\bar{z}' - \frac{i}{2}}{\bar{z}'_0 - \frac{i}{2}} \right)^\eta \left(\frac{\bar{z}' + \frac{i}{2}}{\bar{z}'_0 + \frac{i}{2}} \right)^{\tilde{\eta}} \Phi(\bar{z}_0), \tag{59}$$

must also be so. This requirement restricts $\tilde{\eta}$ to be a nonnegative integer:

$$\tilde{\eta} = -\gamma + \frac{\lambda \varepsilon_n}{\sqrt{m^2 - \varepsilon_n^2}} = n, \quad n = 0, 1, \dots \tag{60}$$

and $\eta = -n - 2\gamma$, from which the energy eigenvalues are seen to be:

$$\frac{\varepsilon_n}{m} = \left\{ 1 + \frac{\lambda^2}{\left(\sqrt{\chi^2 - \lambda^2} + n \right)^2} \right\}^{-1/2}. \tag{61}$$

Thus, as in the nonrelativistic case [4], the bounded spectrum can be determined from the requirement of analyticity on the transform.

Determination of eigenfunctions

From equation (59) and the condition $\Phi(\bar{z}) \rightarrow 0$ for $|\bar{z}| \rightarrow \infty$ (See Lemma 1 of Section 2) it can be seen that:

$$\Phi(\bar{z}) \sim \bar{z}^{-2\gamma}, \text{ for } |\bar{z}| \rightarrow \infty. \quad (62)$$

So, the limit:

$$\lim_{\bar{z}_0 \rightarrow -i\infty} \frac{\Phi(\bar{z}_0)}{\left(\bar{z}_0 - \frac{i}{2}\right)^\eta \left(\bar{z}_0 + \frac{i}{2}\right)^{\tilde{\eta}}} = \phi \quad (63)$$

is finite.

Moreover, for γ as given in equation (53), the matrices A and B can be written as:

$$2\eta A = \left(-\gamma + \chi - \lambda \sqrt{\frac{m + \varepsilon_n}{m - \varepsilon_n}}\right) \left(-\sqrt{\frac{1}{\frac{m - \varepsilon_n}{m + \varepsilon_n}}}\right) \otimes \begin{pmatrix} 1 & \frac{\lambda}{\gamma - \chi} \end{pmatrix}, \quad (64)$$

$$2\tilde{\eta} B = \left(-\gamma + \chi + \lambda \sqrt{\frac{m + \varepsilon_n}{m - \varepsilon_n}}\right) \left(+\sqrt{\frac{1}{\frac{m - \varepsilon_n}{m + \varepsilon_n}}}\right) \otimes \begin{pmatrix} 1 & \frac{\lambda}{\gamma - \chi} \end{pmatrix}. \quad (65)$$

Therefore, up to an overall multiplicative constant:

$$\begin{aligned} \Phi_n(\bar{z}) &= \left(-\gamma + \chi - \lambda \sqrt{\frac{m + \varepsilon_n}{m - \varepsilon_n}}\right) \left(-\sqrt{\frac{1}{\frac{m - \varepsilon_n}{m + \varepsilon_n}}}\right) \\ &\int_{-i\infty}^{\bar{z}} d\bar{z}' \left(\bar{z}' - \frac{i}{2}\right)^{-(n+2\gamma)-1} \left(\bar{z}' + \frac{i}{2}\right)^n + \\ &\left(-\gamma + \chi + \lambda \sqrt{\frac{m + \varepsilon_n}{m - \varepsilon_n}}\right) \left(+\sqrt{\frac{1}{\frac{m - \varepsilon_n}{m + \varepsilon_n}}}\right) \\ &\int_{-i\infty}^{\bar{z}} d\bar{z}' \left(\bar{z}' - \frac{i}{2}\right)^{-(n+2\gamma)} \left(\bar{z}' + \frac{i}{2}\right)^{n-1}. \end{aligned} \quad (66)$$

The integrals in equations (66) can be evaluated on the imaginary negative axis and analytically continued to the half-plane. In this way, one obtains:

$$\Phi_n(\bar{z}) = \left(-\gamma + \chi - \lambda \sqrt{\frac{m + \varepsilon_n}{m - \varepsilon_n}}\right) \left(-\sqrt{\frac{1}{\frac{m - \varepsilon_n}{m + \varepsilon_n}}}\right)$$

$$\begin{aligned}
& (1 + i\bar{z})^{-2\gamma} {}_2F_1 \left(-n, 2\gamma; 2\gamma + 1; \frac{1}{1 + i\bar{z}} \right) + \\
& \left(-\gamma + \chi + \lambda \sqrt{\frac{m + \varepsilon_n}{m - \varepsilon_n}} \right) \left(+ \sqrt{\frac{1}{\frac{m - \varepsilon_n}{m + \varepsilon_n}}} \right) \\
& (1 + i\bar{z})^{-2\gamma} {}_2F_1 \left(-n + 1, 2\gamma; 2\gamma + 1; \frac{1}{1 + i\bar{z}} \right), \tag{67}
\end{aligned}$$

where ${}_2F_1(\dots)$ is a Gauss hypergeometric function.

Notice that this solutions fulfill the hypothesis of Lemma 2, which guarantees that we will obtain all the solutions in the configuration space. In order to do so, the "inverse" transform must be performed. To this end, an explicit $\chi^\gamma(q)$ function must be chosen. For convenience, we adopt:

$$\chi^\gamma(q) = \frac{1}{2\pi\Gamma(\gamma)}. \tag{68}$$

When inserting the first term in equation (67) into equation (17), the integral to be solved is then given by:

$$\begin{aligned}
& \lim_{N \rightarrow \infty} \int_{-N}^N db \int_0^\infty \frac{da}{a^2} a^{\gamma-1/2} \frac{a^{3/2} e^{ibq}}{2\pi\Gamma(\gamma)} (1 + i\bar{z})^{-2\gamma} {}_2F_1 \left(-n, 2\gamma; 2\gamma + 1; \frac{1}{1 + i\bar{z}} \right) \\
& = \frac{1}{2\pi\Gamma(\gamma)} \sum_{k=0}^n (-1)^k \binom{n}{k} \frac{2\gamma}{2\gamma + k} \lim_{N \rightarrow \infty} \int_{-N}^N db e^{ibq} \int_0^\infty da a^{\gamma-1} [1 + ib + a]^{-(2\gamma+k)} \\
& = \frac{1}{\Gamma(2\gamma)} \theta(q) q^{\gamma-1} e^{-q} {}_1F_1(-n, 2\gamma + 1; q), \tag{69}
\end{aligned}$$

where ${}_1F_1(\dots)$ is a degenerate hypergeometric function.

The second term in equation (67) can similarly be inverted (through the replacement $-n \rightarrow -n + 1$ in equation (69)). Thus, the eigenfunctions in the configuration space can be seen to coincide with the well known result (as given, for instance in reference [11]).

4 - Conclusions

In conclusion, we have explored the use of bi-orthogonal basis for continuous wavelet transformations, a generalization which is aimed at relaxing the so-called admissibility condition on the analyzing wavelet, and turns out to be useful for computational reasons.

For definiteness, we have considered the radial dependence of functions in \mathbf{R}^3 . As is well known, choosing as analyzing wavelet the function in equation (2), with $\gamma > 1$, the wavelet transform in equation (5) is an isometry between the Hilbert spaces $\mathbf{L}^2(\mathbf{R}^+, q^2 dq)$ and $\mathcal{B}_{2\gamma-1}$.

In Lemma 1, we have studied the transformation acting on functions $f(q) \in \mathbf{L}_{loc}^1(\mathbf{R}^+, q^\gamma dq) \cap \mathbf{L}^2((1, \infty), dq)$, with $0 < \gamma < 1$, a region where the analyzing wavelet is not admissible and can even be non square integrable. We have shown that the transform $F(\bar{z})$ so defined is an analytic function in the half-plane $Im \bar{z} < 0$, such that $F(\bar{z}) \rightarrow_{|Re z| \rightarrow \infty} 0$, with $Im z = a > 0$, and $F(\bar{z}) \rightarrow_{Im z \rightarrow \infty} 0$, and that the transformation maps differential operators acting on $f(q)$ into differential operators acting on $F(\bar{z})$. Moreover, we have proved that, if $f(q) \in \mathbf{L}^2(\mathbf{R}^+, q^2 dq)$, then $\partial_{\bar{z}} F(\bar{z}) \in \mathcal{B}_{2\gamma+1}$.

In Lemma 2, we have established that - for $F(\bar{z})$ having an asymptotic behaviour as given by equation (16) - the transformation has a right inverse through the use of a bi-orthogonal basis.

In Lemma 3, we have shown that the transformation defined by equation (5), for $0 < \gamma < 1$, is a mapping between a dense subspace of $\mathbf{L}^2(\mathbf{R}^+, q^2 dq)$ and a dense subspace of a pre-Hilbert space \mathcal{A}_γ , which preserves the norm (defined in \mathcal{A}_γ in terms of the scalar product of derivatives in $\mathcal{B}_{2\gamma+1}$).

Finally, as an example of the interest of our results, we have studied the spectrum of relativistic Hydrogen-like atoms. We have shown that, in the determination of eigenvalues of the Hamiltonian of this system and of their associated radial eigenfunctions, a wavelet transformation can be employed, and the calculation is greatly simplified by the choice $\gamma = +\sqrt{\chi^2 - \lambda^2}$. For physical reasons, γ can be any real number greater than zero, which makes apparent the need for our generalization of wavelet transforms. By applying the results proved in our three Lemmas, we have determined the spectrum from the requirement of analyticity on the transform, and we have reconstructed the associated radial eigenfunctions through the use of a bi-orthogonal basis. Both the eigenvalues and eigenfunctions thus obtained can be seen to coincide with standard results.

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